Examination Number:

Set:

Shore

Year 12
HSC Assessment Task 3
Half-Yearly Exam
April 27 2015

Mathematics Extension 1

General Instructions

- Reading time − 5 minutes
- Working time − 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- Answer Questions 1–10 on the Multiple Choice Answer Sheet provided
- In Questions 11–14 show relevant mathematical reasoning and/or calculations
- Start each of Questions 11–14 in a new writing booklet
- Write your examination number on the front cover of each booklet

If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

Total marks – 70

Section I Pages 2–6
10 marks
- Attempt Questions 1−10
- Allow about 15 minutes for this section

Section II Pages 7–12
60 marks
- Attempt Questions 11−14
- Allow about 1 hour 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1–10.

1 The polynomial $2x^3+x-4=0$ has roots α,β and γ. What is the value of $\alpha+\beta+\gamma$?

(A) -2

(B) $-\frac{1}{2}$

(C) 0

(D) 2

2 The point R divides the interval $P\ (3,-6)$ and $Q\ (6,-9)$ externally in the ratio 2:1.

What are the coordinates of R?

(A) $(9,-12)$

(B) $(5,-8)$

(C) $(4,-7)$

(D) $(0,-3)$

3 If $\int_0^k (3x-6)\,dx=0$ and $k\neq 0$, what is the value of k?

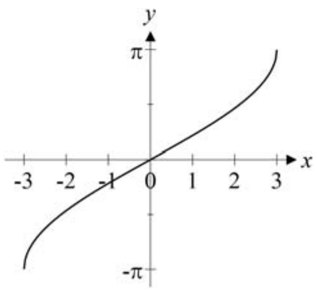
(A) 4

(B) 2

(C) -2

(D) -4

4 What is the equation of the function shown in the graph below?



(A) $y=\frac{1}{2}\sin^{-1}3x$

(B) $y=\frac{1}{2}\sin^{-1}\frac{x}{3}$

(C) $y=2\sin^{-1}\frac{x}{3}$

(D) $y=2\sin^{-1}3x$

5 Which of the following is the Cartesian Equation of the variable point $P(2\cos t, 2\sin t)$, where t is the parameter?

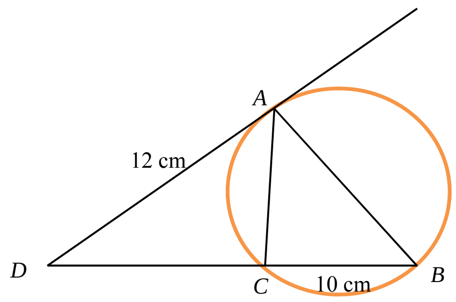
(A) $y=\tan x$

(B) $x^2+y^2=1$

(C) $x^2+y^2=4$

(D) $x^2=4ay$

6 *ABC* is a triangle inscribed in a circle. The tangent to the circle at *A* meets *BC* produced at *D* where *BC* = 10 cm and *AD* = 12 cm. What is the length of *BD*?



NOT TO SCALE

(A) 8

(B) 18

(C) 26

(D) 28

7 Let $t=\tan\frac{\theta}{2}$ where $0<\theta<\pi$. Which of the following gives the correct expression for $\sin\theta-\cos\theta$?

(A) $\frac{2t-1-t^2}{1+t^2}$

(B) $\frac{t^2+2t+1}{1+t^2}$

(C) $\frac{1-t^2-2t}{1+t^2}$

(D) $\frac{t^2+2t-1}{1+t^2}$

8 What is the value of $\cos 2\theta$, given $\cos\theta = \frac{3}{5}$ and $\sin\theta \geq 0$?

(A) $\frac{6}{5}$

(B) $\frac{24}{25}$

(C) $\frac{7}{25}$

(D) $-\frac{7}{25}$

9 Which of the following is an expression for $\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right)$?

(A) $\frac{2}{2+x^2}$

(B) $\frac{2}{4+x^2}$

(C) $\frac{4}{2+x^2}$

(D) $\frac{4}{4+x^2}$

10 Which of the following is the solution to the expression $\int \frac{dx}{\sqrt{9-4x^2}}$?

(A) $\sin^{-1}\frac{2x}{3}+c$

(B) $\sin^{-1}\frac{3x}{2}+c$

(C) $\frac{1}{2}\sin^{-1}\frac{2x}{3}+c$

(D) $\frac{1}{2}\sin^{-1}\frac{3x}{2}+c$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Start each of Questions 11–14 in a new writing booklet.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\lim_{x \to 0} \frac{\sin 5x}{4x}$. **1**

(b) Find $\frac{d}{dx}\left(\frac{\sin 3x}{x}\right)$. **2**

(c) Find the size of the acute angle between the lines $y - 5x - 9 = 0$ and $3y = 2x + 8$. **2**

(d) Find $\int \left(1 + \tan^2(x+1)\right) dx$. **2**

(e) Find in simplest form the exact value of $\cos 15°$. **2**

(f) Solve $\frac{2}{x+3} \geq 1$. **3**

(g) The area of a sector of a circle of radius 9 cm is 75 cm^2. Find the length of the arc of the sector. **3**

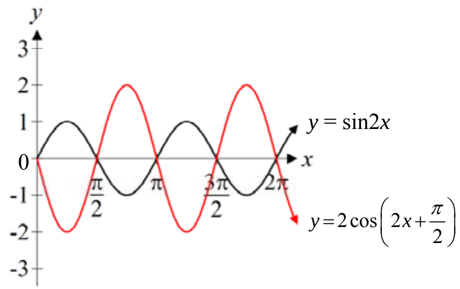
Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Find the values of *a* and *b* that make the polynomial $P(x) = 2x^3 + ax^2 - 13x + b$ exactly divisible by $x^2 - x - 6$. **3**

(b) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \theta)$ where $R > 0$ and $0 < \theta < \frac{\pi}{2}$. **2**

(ii) Hence or otherwise solve the equation $\sqrt{2} = \sin x - \sqrt{3}\cos x$ for $0 \leq x \leq 2\pi$. **2**

(c) The diagram below shows two curves $y = \sin 2x$ and $y = 2\cos\left(2x + \frac{\pi}{2}\right)$. **3**

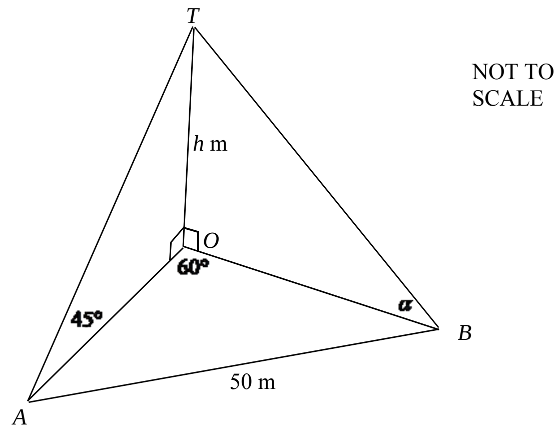


Determine the area enclosed between the curves for the domain $0 \leq x \leq \pi$.

Question 12 continued on page 9

Question 12 (continued)

(d)



In the diagram, the points A, B and O are in the same horizontal plane. A and B are 50 m apart and $\angle AOB = 60^o$. OT is a vertical tower of height h metres. The angles of elevation of T from A and B respectively are 45^o and α. (α is acute).

(i) Explain why $AO = h$. **1**

(ii) Prove that $h^2 \cot^2 \alpha - h^2 \cot \alpha + h^2 = 50^2$ **2**

(iii) Given the tower is 30 m high, find the angle α correct to the nearest degree. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) Find $\dfrac{d}{dx}\left(\left(\log_e(x+4)\right)^3\right)$. **2**

(ii) Hence or otherwise evaluate $\displaystyle\int_{-3}^{0} \frac{\left(\log_e(x+4)\right)^2}{x+4}\,dx$. **2**

(b) A straight line through $T(0,-a)$ cuts the parabola $x^2 = 4ay$ at $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$.

(i) Show that the equation of TP is **2**

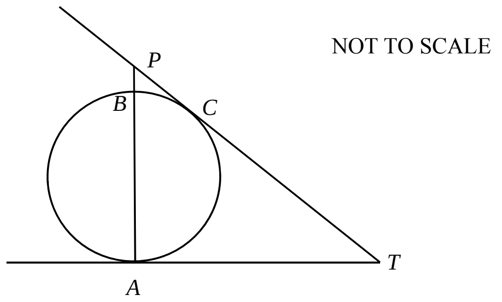
$$2py = x\left(p^2+1\right) - 2ap.$$

(ii) Prove that for TP to pass through Q, $pq = 1$. **2**

(iii) Hence or otherwise prove that $\dfrac{1}{SP} + \dfrac{1}{SQ} = \dfrac{1}{a}$, where S is the focus of the parabola. **3**

Question 13 continued on page 11

(c) In the diagram, AB is a diameter of the circle ABC. The tangents at A and C meet at T. The lines TC and AB are produced to meet at P.



Copy the diagram into your examination booklet.

(i) Prove that $\angle BCP = 90^o - \angle CAT$. **2**

(ii) Explain why $ATCB$ could never be a cyclic quadrilateral. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Use Mathematical Induction to prove that **3**
$1\times 2 + 2\times 2^2 + 3\times 2^3 + ... + n\times 2^n = (n-1)\times 2^{n+1} + 2$ for $n \geq 1$.

(b) Consider the function $f(x) = \dfrac{e^x}{4+e^x}$.

(i) Determine whether $f(x)$ has any stationary points. **3**

(ii) Find the point of inflexion given that **2**
$f''(x) = \dfrac{4e^x(4-e^x)}{(4+e^x)^3}$.

(iii) Show that $0 < f(x) < 1$ for all x. **2**

(iv) Sketch the curve $y = f(x)$. **2**

(v) Explain whether $f(x)$ has an inverse function or not. **1**

(vi) Find the inverse function $y = f^{-1}(x)$. **2**

END OF PAPER

Q1	C	Q6	B
Q2	A	Q7	D
Q3	A	Q8	D
Q4	C	Q9	B
Q5	C	Q10	C

11a) $\lim_{x\to 0} \frac{\sin 5x}{4x} = \frac{1}{4}\lim_{x\to 0} \frac{\sin 5x}{x}$

$= \frac{5}{4}\lim_{x\to 0}\frac{\sin 5x}{5x}$

$= \frac{5}{4}$

b) $\frac{d}{dx}\left(\frac{\sin 3x}{x}\right) = \frac{3x\cos 3x - \sin 3x}{x^2}$

c) $y = 5x+9 \quad m = 5$

$3y = 2x+8 \quad m = \frac{2}{3}$

$\tan\alpha = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\frac{5-\frac{2}{3}}{1+\frac{10}{3}}\right|$

$= 1$

$\alpha = 45°$

d) $\int (1+\tan^2(x+1))\,dx$

$= \int \sec^2(x+1)\,dx$

$= \tan(x+1) + c$

e) $\cos 15° = \cos(45-30)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{1}{\sqrt2} \times \frac{\sqrt3}{2} + \frac{1}{\sqrt2}\times\frac{1}{2}$

$= \frac{\sqrt3}{2\sqrt2} + \frac{1}{2\sqrt2}$

$= \frac{\sqrt6+\sqrt2}{4}$

f) $\frac{2}{x+3} \geq 1 \qquad x \neq -3$

$2(x+3) \geq (x+3)^2$

$0 \geq (x+3)(x+1)$

$-3 < x \leq -1$

g) $\text{Area} = 75\text{ cm}^2 \qquad r = 9$

$A = \frac{1}{2}r^2\theta$

$75 = \frac{1}{2}\times 81 \times \theta$

$\theta = \frac{150}{81}$

$l = r\theta$

$= 9 \times \frac{150}{81}$

$= 16\frac{2}{3}\text{ cm}$

12a) $P(x) = 2x^3 + ax^2 - 13x + b$

$(x-3)$ & $(x+2)$ are factors

$P(3) = 54 + 9a - 39 + b = 0$

$P(-2) = -16 + 4a + 26 + b = 0$

$0 = 10 + 4a + b$

$0 = 15 + 9a + b$

$0 = 5 + 5a$

$a = -1$

$10 - 4 + b = 0$

$b = -6$

b) i) $\sin x - \sqrt3\cos x = R\sin(x-\theta)$

$\sin x - \sqrt3 \cos x = R\sin x\cos\theta - R\cos x\sin\theta$

$R\cos\theta = 1$

$R\sin\theta = \sqrt3$

$R^2 = 1^2 + \sqrt3^2$

$R = 2 \quad R>0$

$\tan\theta = \sqrt3$

$\theta = \frac{\pi}{3}$

$\sin x - \sqrt3\cos x = 2\sin(x-\frac{\pi}{3})$

ii) $\sqrt2 = 2\sin(x - \frac{\pi}{3})$

$\frac{1}{\sqrt2} = \sin(x-\frac{\pi}{3})$

$x - \frac{\pi}{3} = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

$x = \frac{7\pi}{12}$ and $\frac{13\pi}{12}$

c) $A = 2\int_0^{\frac{\pi}{2}} (\sin 2x - 2\cos(2x+\frac{\pi}{2}))\,dx$

$= 2\left[-\frac{1}{2}\cos 2x - \sin(2x+\frac{\pi}{2})\right]_0^{\frac{\pi}{2}}$

$= 2(\frac{1}{2} - -\frac{1}{2}(1 - 1))$

$= 6u^2$

d) i) $AO = h$ as ΔATO is isosceles as $\angle ATO = 45°$ $\therefore AO = OT$

ii) $AO = h \qquad \frac{h}{OB} = \tan\alpha$

$OB = h\cot\alpha$

$a^2 = b^2 + c^2 - 2bc\cos A$

$50^2 = h^2 + h^2\cot^2\alpha - 2xhxh\cot\alpha\cos 60$

$= h^2 + h^2\cot^2\alpha - 2h^2\cot\alpha \times \frac{1}{2}$

$50^2 = h^2 + h^2\cot^2\alpha - h^2\cot\alpha$

iii) $50^2 = 30^2 + 30^2\cot^2\alpha - 30^2\cot\alpha$

$1600 = 900\cot^2\alpha - 900\cot\alpha$

$0 = 9\cot^2\alpha - 9\cot\alpha - 16$

Let $u = \cot\alpha$

$0 = 9u^2 - 9u - 16$

$u = \frac{9 \pm \sqrt{81 - 4\times9\times-16}}{2\times 9}$

$= 1.924 \qquad u>0$

$\tan\alpha = \frac{1}{1.924}$

$\alpha = 27°$

13 a) i) $\frac{d}{dx}\left(\ln(x+4)\right)^3 = \frac{3(\ln(x+4))^2}{x+4}$

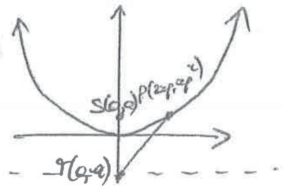
ii) $\int_{-3}^{0} \frac{(\ln(x+4))^2}{x+4}\,dx$

$= \frac{1}{3}\left[(\ln(x+4))^3\right]_{-3}^{0}$

$= \frac{1}{3}\left[(\ln 4)^3 - 0\right]$

$= \frac{1}{3}\left[(\ln 4)^3\right]$

b) i)



PT ⇒ $m = \frac{ap^2+a}{2ap}$

$= \frac{p^2+1}{2p}$

$b = -a$

$y = mx + b$

$y = \frac{p^2+1}{2p}x - a$

$2py = (p^2+1)x - 2ap$

ii) $Q(2aq, aq^2)$ satisfies eqn

$2apq^2 = (p^2+1)2aq - 2ap$

~~$2apq^2 = 2aqp^2 + 2aq - 2ap$~~

$pq^2 - qp^2 = q - p$

$pq(q-p) = q - p$

$pq = 1$.

iii) $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$

SP ⇒ by defn of Parabola

$= ap^2 + a$

$SQ = aq^2 + a$

$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(p^2+1)} + \frac{1}{a(q^2+1)}$

$= \frac{q^2+1+p^2+1}{a(p^2+1)(q^2+1)}$

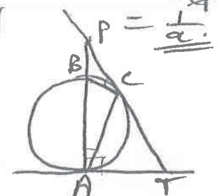
$= \frac{q^2+p^2+2}{a(p^2q^2+p^2+q^2+1)}$

But $pq = 1$ ∴ $p^2q^2 = 1$

$= \frac{q^2+p^2+2}{a(p^2+q^2+2)}$

$= \frac{1}{a}$.

c)



i) ~~∠CAT = ∠ACT~~ ∠BAT = 90° (∠ between ~~TC = AT~~ tangent & radius)

∠BAC = ∠BCP (∠ between tangent & chord = ∠ in alternate segment)

∴ ∠BAC = 90° − ∠CAT

∴ ∠BCP = 90° − ∠CAT

ii) For cyclic quadrilateral

∠BAT + ∠BCT = 90°

but ∠BCA = 90° (∠ in semicircle)

∠BAT = 90°

∴ ∠BCT = $B\hat{C}A$ + ∠ACT

∠BAT + ∠BCT = 90° + 90° + ∠ACT

≠ 180°.

14. a) $1\times2 + 2\times2^2 + \dots + n\times2^n = (n-1)\times2^{n+1} + 2$

(S1) Prove true for $n=1$

$1\times2 = 0\times2^2 + 2$ ✓

True for $n=1$

(S2) Assume true for $n=k$

$1\times2 + 2\times2^2 + \dots + k\times2^k = (k-1)2^{k+1} + 2$

(S3) Prove true for $n=k+1$

$1\times2 + 2\times2^2 + \dots + k\times2^k + (k+1)2^{k+1} = k\cdot2^{k+2} + 2$

From Step 2

$\underbrace{1\times2 + 2\times2^2 + \dots + k\times2^k}$ $= (k-1)2^{k+1} + 2$

$\therefore 1\times2 + 2\times2^2 + \dots + k\times2^k + (k+1)2^{k+1}$

$= (k-1)2^{k+1} + (k+1)2^{k+1} + 2$

$= 2^{k+1}(k-1+k+1) + 2$

$= 2^{k+1}\cdot 2k + 2$

$= k\cdot2^{k+2} + 2$.

(S4) Seeing true for $n=1$, $n=k+1$, assuming true for $n=k$,

let $k=1$ ∴ $n=2$

$k=2$ ∴ $n=3$ etc

∴ true for all integral 'n'.

b) i) $y = \frac{e^x}{4+e^x}$

Stat Pts at $\frac{dy}{dx} = 0$

$y' = \frac{e^x(4+e^x) - e^x \cdot e^x}{(4+e^x)^2}$

$= \frac{4e^x + e^{2x} - e^{2x}}{(4+e^x)^2}$

$0 = 4e^x$

∴ $e^x \neq 0$ ∴ Not Possible.

∴ No Stat Pts.

ii) Possible pt of inflexion at $y'' = 0$

$y'' = \frac{4e^x(4-e^x)}{(4+e^x)^3}$

$0 = 4 - e^x$

$x = \ln 4$. ∴ $y = \frac{4}{4+4}$

$= \frac{1}{2}$.

∴ $(\ln 4, \frac{1}{2})$

Test for pt of inflexion

x	$\ln 3$	$\ln 4$	$\ln 5$
y''	$\frac{12}{7^3}$	0	$\frac{-20}{9^3}$

as sign of y'' changes

∴ $(\ln 4, \frac{1}{2})$ is a pt of inflexion

iii) $x \to \infty$ $e^x \to \infty$

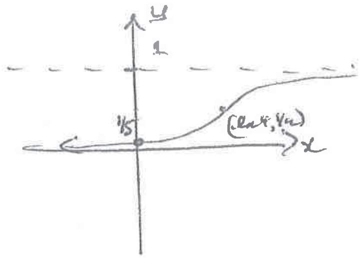
∴ $\lim_{x\to\infty} \frac{e^x}{4+e^x} = \lim_{x\to\infty} \frac{1}{\frac{4}{e^x}+1}$

$= 1$

~~$\lim_{x\to-\infty}$~~ $e^{-x} \to 0$

$\lim_{x\to-\infty} \frac{e^x}{4+e^x} = 0$.

iv)



v) Always monotonic increasing or horizontal line test

vi) $y = \dfrac{e^x}{e^x+4}$

$x = \dfrac{e^y}{e^y+4}$

$xe^y + 4x = e^y$

$(x-1)e^y = -4x$

$e^y = \dfrac{4x}{1-x}$

$y = \ln\left(\dfrac{4x}{1-x}\right)$.